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# Asymptotic Behavior of the Density for Two-Particle Annihilating Exclusion

**V. Belitsky**[1]

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We consider a stochastic process which presents an evolution of particles of two types, $A$ and $B$, on $\mathbb{Z}^d$ with annihilations between particles of opposite types. Initially, at each site of $\mathbb{Z}^d$, independently of the other sites, we put a particle with probability $2\rho < 1$ and assign to it one of two types with equal chances. Each particle evolves on $\mathbb{Z}^d$ in the following manner: independently from the others, it waits an exponential time with mean 1, chooses one of its neighboring sites on the lattice $\mathbb{Z}^d$ with equal probabilities, and jumps to the site chosen. If the site to which a particle attempts to move is occupied by another particle of the same type, the jump is suppressed; if it is occupied by a particle of the opposite type, then both are annihilated and disappear from the system. The considered process may serve as a model for the chemical reaction $A + B \to$ inert. Let $\rho(t)$ denote the density of particles in this process at time $t$. We prove that there exist absolute finite constants $c(d)$ and $C(d)$ such that for all sufficiently large $t$, $c(d)t^{-d/4} \leqslant \rho(t) \leqslant C(d)t^{-d/4}$ in the dimensions $d \leqslant 4$ and $c(d)t^{-1} \leqslant \rho(t) \leqslant C(d)t^{-1}$ in all higher dimensions. This completes and makes more precise the results obtained by us earlier and shows that asymptotically the density behaves like that in a similar process called two-particle annihilating random walks which was studied by Bramson and Lebowitz. Our proofs are based on the approach developed in their and our works. We use the basic properties of random walk and various tools which have been designed to study simple symmetric exclusion processes.

**KEY WORDS**: Diffusion-dominated reaction; two-particle annihilating exclusion; asymptotic upper and lower bounds of the density.

## 1. INTRODUCTION

The process called two-particle annihilating exclusion (abbreviated to AE) was introduced and studied in ref. 2. It was shown there that $\rho(t)$, the

[1] Instituto de Matemática e Estatística, Universidade de São Paulo, CEP 01452-990, São Paulo, SP, Brazil. E-mail: vbel1@ime.usp.br.

density of particles in the AE at time $t$, is asymptotically bounded from above by $Ct^{-1}$ in the dimensions $d>4$ and does not exceed $t^{-d/4}t^{\varepsilon}$ when $t \geqslant t(\varepsilon)$ for all $\varepsilon>0$ in the dimensions $d \leqslant 4$. The present paper provides an asymptotic lower bound for the density of particles in the AE (Propositions 1 and 3 from Sections 3 and 5, respectively) and improves the upper bound calculated in ref. 2 for $d \leqslant 4$ (Proposition 2 from Section 4). When combined, these propositions state the following:

**Theorem 1.** Let $\rho(t)$ denote the density of particles in the two-particle annihilating exclusion at time $t$. Then there exist absolute positive finite constants $c(d)$, $C(d)$ such that

$$c(d)\, t^{-d/4} \leqslant \rho(t) \leqslant C(d)\, t^{-d/4} \qquad \text{when} \quad d \leqslant 4$$

$$c(d)\, t^{-1} \leqslant \rho(t) \leqslant C(d)\, t^{-1} \qquad \text{when} \quad d \geqslant 4$$

for all sufficiently large $t$.

The proof of Theorem 1 uses essentially the methodology developed by Bramson and Lebowitz[3] for studying the asymptotic behavior of density in a process similar to the AE which is called two-particle annihilating random walks (ARW). In the ARW, particles of two types, say $A$ and $B$, evolve on $\mathbb{Z}^d$. Each particle executes a (simple symmetric) random walk independently of all other particles and is annihilated and removed from the process when it meets an opposite-type particle; the latter is annihilated as well. The AE is a modification of the ARW obtained by imposing one additional condition: when a particle attempts to move to a site which is occupied by another particle of the same type, this move is suppressed. Thus, in the AE, two particles of the same type never occupy the same site simultaneously. Consequently, we expect the AE to be a more appropriate model of the chemical reaction $A+B \to \text{inert}$ than the ARW (for the relation of the considered processes to chemistry, see Section 1 of ref. 3 and the references therein). Our interest in the AE was motivated by this fact. Also, the upper bound for $\rho(t)$ provided in Theorem 1 is used in ref. 4 for studying the occurrence of a rare event in the exclusion process.

The particles of the same type interact in the AE by the rules of the (simple symmetric) exclusion process (see Chapter VIII of ref. 6 for the definition). In our proofs, we substitute the exclusion process by another process called a stirring system. It is constructed in the following way: to each bond of the lattice $\mathbb{Z}^d$ we attach an alarm clock such that the times when its alarm goes off form a Poisson point process on $[0, \infty)$ with the intensity $(2d)^{-1}$; each time when the alarm goes off at a bond, the contents of the sites connected by this bond interchange (see ref. 5 for the complete definition). It is known that if a simple symmetric exclusion process and a stirring system start from the same configuration of particles then the distribution of this ensemble of particles is the same in both processes. The advantage of the stirring system is that the marginal motion of a single particle coincides in law with a simple symmetric random walk. This property made it possible to adapt to the AE the majority of the tools which had been designed for the ARW in ref. 3.

Let $\rho_{\text{arw}}(t)$ denote the density of particles in the ARW at time $t$. Comparing the results of ref. 3 to the above theorem, one sees that $\rho_{\text{arw}}(t)$ and $\rho(t)$ have the same asymptotic exponents [though $c(d)$ and $C(d)$ may differ]. This coincidence for $d \geqslant 4$ has the following intuitive ground. In these dimensions, the exponent of the decay of $\rho_{\text{arw}}(t)$ is the same as the one obtained by the mean-field approximation argument (see Section 1 in ref. 3). Thus, because of the property of the stirring system mentioned in the above paragraph, this argument pertains also to the case of the AE, leading to the same result as for the ARW. Though the rigorous derivation of the decay of particle density in the ARW made in ref. 3 is much more sophisticated than just the mean-field approximation, we did not have difficult technical problems in adapting it for the AE in $d > 4$. One may see, for example, that Section 5, which gives the lower bound for $d > 4$, is in fact a repetition of the argument of Bramson and Lebowitz for these dimensions with slight corrections that made it suitable for the AE.

The methodology employed for $d \leqslant 4$ is completely different from that in $d > 4$. Both for the AE in this paper and for the ARW in ref. 3 the correct lower bound for the density in $d \leqslant 4$ is obtained by evaluating the advantage of particles of one type over another in a cube $D$ of side $R$ at time 0 and showing then that this advantage remains essentially of the same order at time $\sqrt{R}$. The implementation of this program requires us to control the motion of particles in a way more precise than the one provided just by the mean-field approximation. We will now present roughly the method which gives the desired control. This will allow us to demonstrate where and why the technique developed by Bramson and Lebowitz fails to work for the AE and to indicate the basic ingredients of the technique which we substitute for it in the present paper.

We will consider two processes called MARW and MAE which are modifications of the ARW and AE, respectively. They are defined as follows. Let us call "alive" any particle that has not been annihilated. We postulate that alive particles move in the MARW and MAE exactly in the same manner as they do in, respectively, the ARW and AE. However, in the modified processes, an alive particle will not be discarded immediately after its annihilation. Instead, it will be marked "dead" and continue to evolve according to the stirring rule in the MAE and to execute an independent random walk in the MARW. We also postulate that a dead particle cannot annihilate an alive particle of the opposite type. This ensures that the distributions of the alive particles in a process and in its modification are the same. We note that the initial configuration of particles for the AE (in this paper) and for the ARW (in ref. 3) are chosen in such a way that the distributions of all the particles of the same type (i.e., regardless the labels "alive" or "dead") in the MAE and the MARW do not change in time. Thus, the distribution of the alive particles in a process may be characterized if we know that of the dead particles in the modified version. The evolution of the dead particles possesses nice properties: (i) new dead particles appear in pairs, one $A$ and one $B$ particle, simultaneously and at the same site of $\mathbb{Z}^d$, and (ii) in the case of the MARW, the motions of the particles from the same pair after their appearance coincide in law and do not depend on anything else. These properties were essentially used in ref. 3 to find the decay of $\rho_{\mathrm{arw}}(t)$. Unfortunately, (ii) is not true for the MAE: due to the stirring rules, when an alive particle moves to the place occupied by a dead particle of the same type, it will pull the latter to its former position. Thus, in the MAE, the evolution of dead particles is dependent on that of the alive ones. The measure of this dependence which we needed in order to establish the decay of $\rho(t)$ in $d \leqslant 4$ is related to

$$\mathbb{E}_\eta\left[I_{\{x_t \in D\}} \sum I_{\{y_t \in D\}}\right] - \mathbb{E}[I_{\{Z_t^x \in D\}}]\, \mathbb{E}_\eta\left[\sum I_{\{y_t \in D\}}\right] \tag{1}$$

where $t$ is time; $D$ is a cube in $\mathbb{Z}^d$ with side $\sqrt{t}$; $z_t$ denotes the position at time $t$ in the stirring system of the particle which originated from $z \in \mathbb{Z}^d$; $\mathbb{E}_\eta$ is the expectation with respect to the law of the stirring process with $\eta$ being the initial configuration; $x$ is assumed to contain an $A$ particle in $\eta$; both sums are taken over all those $y$'s which contain $A$ particles in $\eta$; and $Z_s^x$, $s \geqslant 0$, is a simple symmetric random walk starting from $x$ which is independent of the stirring process. Observe that if we substitute $\sum I_{\{y_t \in D\}}$ by just $I_{\{y_t \in D\}}$ for some $y \neq x$, then the Liggett inequality (ref. 6, Lemma 4.12, Chapter VIII) says (1) is not positive. Andjel[1] modified Liggett's argument and obtained a quantitative estimate on (1) for this case. We used the ideas of Andjel. We showed the sum of (1) over all $x \in \mathbb{Z}^d$ which contain an $A$ particle in $\eta$ does not exceed const $\times \sqrt{t} \times$ (cardinality of the boundary of $D$) independently of $\eta$. Our reasoning is presented in Section 3. This section also derives the lower bound for the dimensions $d < 4$ (the case $d \geqslant 4$ is treated in Section 5). The technique developed in Section 3 allows us to sharpen some intermediate estimates from ref. 2 which leads to improving the final result provided in that paper—the asymptotic upper bound for $\rho(t)$ in the dimensions $d \leqslant 4$. Section 4 describes the modifications one has to make in the reasoning of ref. 2 to obtain the improved bound.

## 2. BASIC DEFINITIONS

In this section we present the studied process and its modification and introduce notations which will have the same meaning throughout the paper. The presentation is brief but should be sufficient for a reader who is acquainted with the general framework of interacting particle systems. For details, we refer to ref. 2.

*Remark on Notation.* $f(x)[\alpha]$ means the value of a random function $f$ attained at the point $x$ and on the element $\alpha$ from the probability space on which this function is defined; we will omit $[\alpha]$ in the above notation when it causes no confusion.

The space of states for the processes which we investigate in this paper will be constructed stemming from the space $\mathscr{Z} := \{0, 1\}^{\mathbb{Z}^d}$ or $\mathscr{X} := \{A, B, A \cup B, 0\}^{\mathbb{Z}^d}$. An element from these spaces is called a configuration of particles, which all are of the same type in the case $\mathscr{Z}$, and may be of two different types, called $A$ and $B$, in the case $\mathscr{X}$. For $\chi \in \mathscr{Z}$ (resp., $\chi \in \mathscr{X}$), we say that $\chi$ has a particle (resp., an $A$ particle, a $B$ particle, both $A$ and $B$ particles) at a site $z \in \mathbb{Z}^d$ if $\chi(z) = 1$ (resp., $A$, $B$, $A \cup B$), and we say $z$ is empty in $\chi$ if $\chi(z) = 0$.

By $(\Omega^A, \mathscr{F}^A, \mu^A)$ and $(\Omega^B, \mathscr{F}^B, \mu^B)$ we denote two independent copies of the probability space of percolation substructures which generate a stirring system on $\mathbb{Z}^d$ (ref. 5 describes how an interacting particle system is constructed using percolation substructures). They will determine the motions of $A$ and $B$ particles respectively. We let ${}^A\eta_t^x[\omega^A]$ designate the position of an $A$ particle in the percolation substructure $\omega^A \in \Omega^A$ at time $t \geqslant 0$, given it originated from $x \in \mathbb{Z}^d$. We define ${}^B\eta_t^x[\omega^B]$ similarly, for a $B$ particle. On the space $(\Omega, \mathscr{F}, \mu) := (\Omega^A \times \Omega^B, \mathscr{F}^A \times \mathscr{F}^B, \mu^A \times \mu^B)$ we then define a process $\eta_t$, $t \geqslant 0$, in the following manner: if $\eta_0 \in \mathscr{X}$ is its initial configuration, then the state at time $t \geqslant 0$ on $\omega = \omega^A \times \omega^B \in \Omega$ is

$$\eta_t[\omega] = \bigcup_{x : \eta_0(x) = A} {}^A\eta_t^x[\omega^A] \cup \bigcup_{x : \eta_0(x) = B} {}^B\eta_t^x[\omega^B] \in \mathscr{X} \tag{2}$$

For each $\eta_0 \in \mathscr{Y} := \{A, B, 0\}^{\mathbb{Z}^d} \subset \mathscr{X}$ and each $\omega \in \Omega$, we then adopt the following procedure: we label "alive" each particle which is present in $\eta_0$; then, for each particle, we change its label to "dead" at the time when it meets (in $\eta_{\cdot}[\omega]$) an alive particle of the opposite type; the latter is called its *annihilating companion*; certainly, it also changes its label to "dead." By

$\tau(x) = \tau(x)[\eta_0, \omega]$ we denote the time when the particle which started from $x$ in $\eta_0$ changed its label in $\omega$. We call $\tau(x)$ *the annihilation time.* The choice of the term "annihilation" was motivated by the process $\xi_\bullet$ defined below in which the dead particles are discarded.

Using the concept of the annihilation time, we define a new process denoted by $\xi_t$, $t \geqslant 0$, in the following manner: the initial configuration is the random variable $\xi_0$ which takes its values in $\mathcal{Y}$ and whose distribution obeys the following rules:

$$\{\xi_0(x)\}_{x \in \mathbb{Z}^d} \text{ is a set of i.i.d. random variables}$$

$$\text{such that } \mathbb{P}[\xi_0(x) \neq 0] = 2\rho \text{ [for } \rho \in (0, 1/2)]$$

$$\text{and } \mathbb{P}[\xi_0(x) = A \mid \xi_0(x) \neq 0] = \mathbb{P}[\xi_0(x) = B \mid \xi_0(x) \neq 0] = 1/2 \quad (3)$$

The dynamics of $\xi_\bullet$ is determined by $\omega \in \Omega$ as follows: for each $t \geqslant 0$,

$$\xi_t[\omega] := {}^A\xi_t[\omega] \cup {}^B\xi_t[\omega]$$

where

$$ {}^A\xi_t[\omega] := \bigcup_{\substack{x : \xi_0(x) = A, \\ \tau(x)[\xi_0, \omega] > t}} {}^A\eta_t^x[\omega] \qquad \text{and} \qquad {}^B\xi_t[\omega] := \bigcup_{\substack{x : \xi_0(x) = B, \\ \tau(x)[\xi_0, \omega] > t}} {}^A\eta_t^x[\omega]$$

In words, $\xi_t[\omega]$ contains those and only those particles which are alive in $\eta_t[\omega]$. We call $\xi_t$, $t \geqslant 0$, *two-particle annihilating exclusion* or simply *annihilating exclusion.* The present paper studies the behavior as $t \to \infty$ of the particle density $\rho(t)$ in $\xi_\bullet$:

$$\rho(t) = \mathbb{P}[\text{the site 0 is occupied by an } A\text{-type particle at time } t \text{ in } \xi_\bullet]$$

$$= \mathbb{P}[\text{the site 0 is occupied by a } B\text{-type particle at time } t \text{ in } \xi_\bullet]$$

To obtain information about $\xi_\bullet$, we will mainly study $\eta_\bullet$ under the same initial distribution of particles as in $\xi_\bullet$. Thus, we make the following modification in notations: if not indicated to the contrary, $\eta_\bullet$ is the process defined by (2) and such that $\eta_0$ coincides in distribution with $\xi_0$ defined in (3). The law of $\eta_t$, $t \geqslant 0$, with a given $\eta_0$ is denoted by $\mathbb{P}_{\eta_0}$; the corresponding mathematical expectation is denoted by $\mathbb{E}_{\eta_0}$. We will usually write $\eta_t^x$ meaning the position in $\eta_\bullet$ at time $t$ of the particle which was initially at $x \in \mathbb{Z}^d$. Since $\eta_0$ has at most one particle per site, it will be always clear if this sign stands for ${}^A\eta_t^x$ or for ${}^B\eta_t^x$.

Finally, for every $\eta_0 \in \mathscr{Y}$, $t \geqslant 0$, and $\omega \in \Omega$, we define the function $k(\cdot) = k(\cdot; \eta_0, t, \omega)$ from $\mathbb{Z}^d$ to $\mathbb{Z}^d$ by the following rule: if $\eta_0(y) \neq 0$ and $\tau(y)[\eta_0, \omega] \leqslant t$, then $k(y)$ denotes the initial position of the annihilating companion of the particle which was initially at $y$; if $\eta_0(y) \neq 0$ and $\tau(y)[\eta_0, \omega] > t$, then $k(y) := y$; if $\eta_0(y) = 0$, then $k(y)$ is not defined.

In the view of the above definition, the notation $\eta_t^{k(y)}$ means the following: it is defined for $y \in \mathbb{Z}^d$ and $\eta_0$ such that $\eta_0(y) \neq 0$ and if $\eta_0(y) = A$ [in the opposite case, interchange $A$ and $B$ in (4)]

$$\eta_t^{k(y)}[\omega] = \begin{cases} {}^A\eta_t^y[\omega] & \text{if } \tau(y)[\eta_0, \omega] > t \\ {}^B\eta_t^z[\omega] & \text{if } \tau(y)[\eta_0, \omega] \leqslant t \end{cases} \qquad \omega \in \Omega, \quad t \geqslant 0 \tag{4}$$

where $z := k(y; \eta_0, t, \omega)$ is the initial position of the annihilating companion of the particle which started from $y$. The sign $\eta_t^{k(y)}$ will be frequently used in the sequel.

## 3. LOWER BOUND FOR DIMENSIONS $d \leqslant 4$

In this section we demonstrate the following result:

**Proposition 1.** There exists an absolute positive constant $c(d)$ such that for large $t$,

$$\rho(t) \geqslant c(d)\, t^{-d/4}$$

*Proof.* Let $D$ denote the cube in $\mathbb{Z}^d$ of side $R_t := \delta t^{1/2}$ centered at the origin, where $\delta$ will be chosen later exclusively in accordance with the values of certain absolute constants. The dependence on these constants leaves enough freedom to choose $\delta$ in such a way that $|D| = (\text{side of } D)^d$ always holds ($|D|$ denotes the number of the sites of $\mathbb{Z}^d$ contained in $D$).

Define

$$I_D(x) := \begin{cases} 1 & \text{if } x \in D \\ 0 & \text{if } x \notin D \end{cases} \qquad \forall x \in \mathbb{Z}^d$$

From the construction of the processes $\eta_\cdot$ and $\xi_\cdot$, we have the following decomposition:

$$\sum_{x : \eta_0(x) = \cdot} I_D(\xi_t^x) = \sum_{x : \eta_0(x) = \cdot} I_D(\eta_t^x) - \sum_{x : \eta_0(x) = \cdot} I_D(\eta_t^x) I_{\{\tau(x) \leqslant t\}} \tag{5}$$

where $\cdot$ may be either $A$ or $B$ throughout in (5). The above decomposition and the inequality $|a| \geqslant |a+b| - |b|$, $a, b \in \mathbb{R}$, yield that

$$
\begin{aligned}
E_0 := \mathbb{E}\left[\left|\sum_{x:\eta_0(x)=A} I_D(\xi_t^x) - \sum_{y:\eta_0(y)=B} I_D(\xi_t^y)\right|\right] \\
\geqslant \mathbb{E}\left[\left|\sum_{x:\eta_0(x)=A} I_D(\eta_t^x) - \sum_{y:\eta_0(y)=B} I_D(\eta_t^y)\right|\right] \\
- \mathbb{E}\left[\left|\sum_{x:\eta_0(x)=A} I_D(\eta_t^x) I_{\{\tau(x)\leqslant t\}} - \sum_{y:\eta_0(y)=B} I_D(\eta_t^y) I_{\{\tau(y)\leqslant t\}}\right|\right] \qquad (6)
\end{aligned}
$$

Let us first estimate from below the first expectation in the right-hand side of (6). It is known (and is easy to verify using graphical representation methods) that if a stirring system and an exclusion process start from the same initial state, then the configuration of particles at each time $t > 0$ is distributed identically for both processes. (Certainly, the trajectory of each single particle differs for these two process, but if we do not distinguish between particles, then the set of sites of $\mathbb{Z}^d$ occupied by all the particles is a random set whose distibution does not depend on whether the particles interacted by the stirring rule or by the exclusion one.) From the construction of the initial state for the process $\eta_\cdot$, the distribution of particles of the same type on $\mathbb{Z}^d$ in $\eta_0$ is the Bernoulli product measure with the density $\rho$. Since this measure is invariant for the exclusion process (see Chapter VIII of ref. 6 for the proof), then the number of particles of the same type which are present in $D$ at time $t > 0$ is distributed identically to that at time 0. Thus, recalling that the $A$ and $B$ particles evolve independently in $\eta_\cdot$, we derive that the first expectation in the right-hand side of (6) equals

$$
\mathbb{E}[|\#\{x \in D: \eta_0(x) = A\} - \#\{y \in D: \eta_0(y) = B\}|] = \mathbb{E}\left[\left|\sum_{x\in D} \theta_x\right|\right] \qquad (7)
$$

where for each $x \in \mathbb{Z}^d$ we defined that $\theta_x$ equals 1 if $\eta_0(x) = A$, $-1$ if $\eta_0(x) = B$, and 0 if $x$ is empty in $\eta_0$. Due to the construction of $\eta_0$, the random variables $\theta_x$, $x \in \mathbb{Z}^d$, are i.i.d. with $\mathbb{P}[\theta_x = 1] = \mathbb{P}[\theta_x = -1] = \rho$. Now, to $X := \sum_{x\in D} \theta_x$ we apply the reasoning used in the proof of Lemma 2.3 of ref. 3 and we get that for an appropriate $c_1$

$$
\mathbb{E}\left|\sum_{x\in D} \theta_x\right| \geqslant c_1 \sqrt{\rho}\, (|D|)^{1/2} = c_1 \sqrt{\rho}\, (R_t)^{d/2} \qquad \text{for all large enough } t \qquad (8)
$$

[The reasoning goes in the following way: when $t$ is large, $X/(\operatorname{Var} X)^{1/2}=X/(2\rho\,|D|)^{1/2}$ is approximately standard normal, thus $\mathbb{P}[X>(2\rho\,|D|)^{1/2}/2]>c$ for appropriate $c>0$ from which (8) follows.] Thus, we conclude that

$$\mathbb{E}\left[\left|\sum_{x:\eta_0(x)=A} I_D(\eta_t^x)-\sum_{y:\eta_0(y)=B} I_D(\eta_t^y)\right|\right]$$
$$\geqslant c_1\sqrt{\rho}\,\delta^{d/2}t^{d/4} \qquad \text{for all large enough } t \tag{9}$$

Next, we will estimate from above the second expectation in the right-hand side of (6). Since particles annihilate in pairs, then using the sign $\eta_t^{k(\cdot)}$ defined in (4), we write down the following identity, which holds for all $\eta_0\in\mathscr{Y}=\{A,B,0\}^{\mathbb{Z}^d}$, all $\omega\in\Omega$, and all $t\geqslant 0$:

$$\sum_{x:\eta_0(x)=A} I_D(\eta_t^x)I_{\{\tau(x)\leqslant t\}}-\sum_{y:\eta_0(y)=B} I_D(\eta_t^y)I_{\{\tau(y)\leqslant t\}}$$
$$=\sum_{x:\eta_0(x)=A}[I_D(\eta_t^x)-I_D(\eta_t^{k(x)})] \tag{10}$$

The mathematical expectation of the square of the right-hand side of (10) equals

$$\begin{aligned}E_1+E_2-E_3:=&\,\mathbb{E}\left(\sum_{\substack{x,y:\eta_0(x)=A,\\ \eta_0(y)=A,\,y\neq x}}[I_D(\eta_t^x)-I_D(\eta_t^{k(x)})][I_D(\eta_t^y)-I_D(\eta_t^{k(y)})]\right)\\ &+\mathbb{E}\left(\sum_{x:\eta_0(x)=A}[I_D(\eta_t^x)-I_D(\eta_t^{k(x)})]\,I_D(\eta_t^x)\right)\\ &-\mathbb{E}\left(\sum_{x:\eta_0(x)=A}[I_D(\eta_t^x)-I_D(\eta_t^{k(x)})]\,I_D(\eta_t^{k(x)})\right)\end{aligned}$$

Based on Lemma 2.2 of ref. 2, one easily finds that $E_1\leqslant 0$. Next, using once more the fact that particles annihilate in pairs, we have that

$$E_3=-\mathbb{E}\left(\sum_{z:\eta_0(z)=B}[I_D(\eta_t^z)-I_D(\eta_t^{k(z)})]\,I_D(\eta_t^z)\right)$$

Thus, the symmetry between $A$ and $B$ particles in the process $\eta_\cdot$ yields that $E_3=-E_2$. As for $E_2$, we remark that it equals the sum of the mathematical expectations from (39) and (41) below in the text. Using Lemmas 3 and 4 for estimating this sum and the Jensen inequality, $\mathbb{E}\{[(\cdot)^2]^{1/2}\} \leqslant \{\mathbb{E}[(\cdot)^2]\}^{1/2}$, we then conclude that (below, we use that $|\Delta_1 D| = 4R_t^{d-1}$)

$$\begin{aligned}\mathbb{E}\left[\left|\sum_{x:\eta_0(x)=A} I_D(\eta_t^x) I_{\{\tau(x)\leqslant t\}} - \sum_{y:\eta_0(y)=B} I_D(\eta_t^y) I_{\{\tau(y)\leqslant t\}}\right|\right] \\ \leqslant (E_1+E_2-E_3)^{1/2} \\ \leqslant [2C\sqrt{t}\,(4R_t^{d-1})]^{1/2} = C_1 \delta^{(d-1)/2} t^{d/4} \end{aligned} \tag{11}$$

Now, (6), (9), and (11) guarantee that we can choose $\delta$ in such a way that

$$E_0 \geqslant c_2 t^{d/4} \qquad \text{for an appropriate } c_2 \text{ when } t \text{ is large enough} \tag{12}$$

Proposition 1 stems from (12) in the following way:

$$\rho(t) = \frac{1}{2\,|D|}\,\mathbb{E}\left[\sum_{x:\eta_0(x)=A \text{ or } B} I_D(\xi_t^x)\right] \geqslant \frac{E_0}{2\,|D|} = \frac{E_0}{2(\delta R_t)^d} \geqslant c_3 t^{-d/4}$$

for all large enough $t$, where $c_3 > 0$ is an absolute constant which may depend on $d$. ∎

In establishing Proposition 1 we used Lemma 3, which is formulated at the end of this section, and Lemma 4 from Section 4. We now proceed to a chain of auxiliary constructions and assertions which will lead to the proof of these lemmas.

Let $\hat{X} := \{0, 1\}^{\mathbb{Z}^d} \times \mathbb{Z}^d$ be the set of all the configurations of particles on $\mathbb{Z}^d$ in which (i) one particles is marked, (ii) no more than one unmarked particle is allowed at a site of $\mathbb{Z}^d$, and (iii) the marked particle may occupy a site which contains an unmarked particle. For $\chi \in \{0, 1\}^{\mathbb{Z}^d}$ and $y \in \mathbb{Z}^d$, we denote by $(\chi; y)$ the element of $\hat{X}$ which we interpret as follows: $\chi$ is the configuration of the unmarked particles and $y$ is the position of the marked one. We denote by $(\chi^{uv}; y)$ the element of $\hat{X}$ which is obtained from $(\chi; y)$ by interchanging the values of $\chi$ at the sites $u$ and $v$, where $u, v \in \mathbb{Z}^d$, $u \neq v$.

Let $U(t)$, $t \geqslant 0$, and $U$ denote, respectively, the semigroup and the generator of the interacting particle system which evolves in $\hat{X}$ according to the following rules: the unmarked particles interact between themselves due to the stirring mechanism, while the marked particle executes a simple symmetric random walk in $\mathbb{Z}^d$ which is independent of the stirring. Formally, $U$ is written as

$$\begin{aligned} Ug(\phi; y) = \sum_{x,z:x\sim z} \frac{1}{2d}[g(\phi^{xz}; y) - g(\phi; y)] \\ + \sum_{x:x\sim y} \frac{1}{2d}[g(\phi; x) - g(\phi; y)], \qquad (\phi; y) \in \hat{X} \end{aligned} \tag{13}$$

where $a \sim b$ means $a, b$ are neighboring sites of $\mathbb{Z}^d$, the first sum is taken over all unordered pairs of neighbors, and $g$ should be from the domain of the definition of $U$ [see the text immediately after (26)].

Set now $X^* := \{(\chi; y) \in \hat{X}\colon \chi(y) = 1\}$, that is, $X^*$ is the set of all elements of $\hat{X}$ in which the site occupied by the marked particle contains necessarily an unmarked particle. By $V(t)$, $t \geqslant 0$, and $V$ we denote, respectively, the semigroup and the generator of the interacting particle system which evolves in $X^*$ according to the following rules: the unmarked particles interact between themselves by the stirring mechanism and the marked particle moves together with the unmarked particle which was initially at its site. Formally, $V$ has the following form:

$$Vg(\phi; y) = \sum_{\substack{x, z : x \sim z \\ x \neq y,\, z \neq y}} \frac{1}{2d} [g(\phi^{xz}; y) - g(\phi; y)]$$

$$+ \sum_{x : x \sim y} \frac{1}{2d} [g(\phi^{xy}; x) - g(\phi; y)], \qquad (\phi; y) \in X^* \tag{14}$$

where the first sum is taken over all unordered pairs of neighbors which satisfy the indicated condition.

The lemma below compares the values of a particular functional of the processes $U$ and $V$. Below, $Z_t^x$, $t \geqslant 0$, denotes the simple symmetric random walk on $\mathbb{Z}^d$ starting from $x$.

**Lemma 1.** For an arbitrary $h \in \mathbb{N}$, let $D := [-h, h]^d \cap \mathbb{Z}^d$ be the cube in $\mathbb{Z}^d$ of side $2h$ centered at the origin. Define

$$\Delta D := \{z = (z^1, \ldots, z^d) \in \mathbb{Z}^d\colon \|z\| := \max_{1 \leqslant i \leqslant d} |z^i| \text{ equals either } h \text{ or } h+1\}$$

to be the union of the boundary points of $D$ and $\mathbb{Z}^d \backslash D$. Let $\Delta_i D$ denote the union of the faces of $\Delta D$ which are orthogonal to the $i$th unit vector $e_i := (0, \ldots, 0, 1, 0, \ldots, 0)$ (1 on the $i$th place):

$$\Delta_i D := \{z \in \Delta D\colon z^i \in \{-h-1, -h, h, h+1\}\}$$

Then, for the function $f\colon \hat{X} \to \mathbb{R}$ of the form

$$f(\chi; y) := \left( \sum_{x : \chi(x) = 1} I_D(x) \right) I_D(y), \qquad (\chi; y) \in \hat{X} \tag{15}$$

any configuration $(\chi; v) \in X^*$, and any $s \geqslant 0$, it holds that

$$0 \leqslant (V(s) - U(s)f)(\chi; v) \leqslant \frac{1}{d} \sum_{j=1}^{d} \int_0^s \sum_{y \in \mathbb{Z}^d} \mathbb{P}[Z_{s-r}^{v} = y](\mathbb{P}[Z_r^y \in \Delta_j D])^2 \, dr \tag{16}$$

*Proof.* We start with some auxiliary constructions which we will use to couple the processes $U$ and $V$. Let $(\Omega^A, \mathscr{F}^A, \mu^A)$ be an independent copy of the probability space of percolation substructures which generates the stirring system (we use the same symbols, since this copy will be used solely within the proof of this lemma). Also, by $(\Gamma, \mathscr{F}^\Gamma, \mu_\Gamma)$ we will denote the probability space of the percolation substructures which generates the simple symmetric random walk on $\mathbb{Z}^d$ which starts at zero. We postulate that $(\Omega^A, \mathscr{F}^A, \mu^A)$ and $(\Gamma, \mathscr{F}^\Gamma, \mu_\Gamma)$ are independent. The mathematical expectation with respect to the measure $\mu_\Gamma$ $(\mu^A)$ will be denoted by $\mathbb{E}_\Gamma$ (respectively, $\mathbb{E}_\Omega$). The expectation with respect to $\mu^A \times \mu_\Gamma$ on the space $(\Omega^A \times \Gamma, \mathscr{F}^A \times \mathscr{F}^\Gamma)$ will be denoted by $\mathbb{E}_{\Omega \times \Gamma}$.

Assume $\omega \in \Omega^A$, $\gamma \in \Gamma$, and $r \geqslant 0$. Put a configuration $\phi \in \{0, 1\}^{\mathbb{Z}^d}$ at time 0 in $\omega$. The configuration which is obtained from $\phi$ at time $r$ in $\omega$ will be designated by $\phi_r^\omega$. Similarly, put a particle at time 0 at the origin of $\mathbb{Z}^d$ in $\gamma$. The position of this particle at time $r$ in $\gamma$ will be designated by $0_r^\gamma$. For $y \in \mathbb{Z}^d$, we then define $y_r^\gamma := 0_r^\gamma + y$, so that $y_r^\gamma$, $r \geqslant 0$, is a trajectory of a simple symmetric random walk in $\mathbb{Z}^d$ starting from $y$. Also, by $(\phi; y)_r^{\omega,\gamma}$, we will designate the configuration $(\varphi; z) \in \hat{X}$ such that $\varphi = \phi_r^\omega$ and $z = y_r^\gamma$.

Below, we will need one important property of the stirring system, which we recall now. Let $\hat{\zeta}, \zeta \in \{0, 1\}^{\mathbb{Z}^d}$ and $z \in \mathbb{Z}^d$ be arbitrary and such that $\zeta(z) = 1 - \hat{\zeta}(z) = 0$, while $\zeta(u) = \hat{\zeta}(u)$ for all $u \neq z$. With some abuse of notation, let $\{z\}$ denote the configuration which has a single particle at $z$ and all other sites empty. Then for any $\omega \in \Omega$ and $r \geqslant 0$ it holds that

$$\hat{\zeta}_r^\omega = \zeta_r^\omega \cup \{z\}_r^\omega \tag{17}$$

In words, $\hat{\zeta}_r^\omega$ has a particle at each site where $\zeta_r^\omega$ has and one more particle which is that very one which originated from $z$; moreover, its position at time $r$ depends solely on $\omega$. One can verify (17) straightforwardly.

Let now $(\phi; y)$ be an arbitrarily fixed configuration from $X^*$ [recall, by the definition, this implies $\phi(y) = 1$] such that $\phi(y + e_1) = 0$ and denote $x := y + e_1$. For this $(\phi; y)$, define the function $\Phi_{(\phi; y)}(\cdot, \cdot, \cdot): \Omega^A \times \Gamma \times \mathbb{R}^+ \to \mathbb{R}$ by

$$\Phi_{(\phi; y)}(\omega, \gamma, r) := f[(\phi; y)_r^{\omega,\gamma}] - f[(\phi^{xy}; y)_r^{\omega,\gamma}] + f[(\phi^{xy}; x)_r^{\omega,\gamma}] - f[(\phi; x)_r^{\omega,\gamma}] \tag{18}$$

where $f$ is from (15).

Assume that $\gamma_1 \in \Gamma$ is such that $y_r^{\gamma_1} \in D$ and $x_r^{\gamma_1} \notin D$. Then, for this $\gamma_1$ and for every $\omega \in \Omega^A$, the last two terms of (18) are zeros because of the form of $f$. Now, due to our choice of $\phi$ and $y$, it holds that $\phi^{xy}(x) = 1 - \phi^{xy}(y) = 1 - \phi(x) = \phi(y) = 1$ and $\phi^{xy}(z) = \phi(z)$ when $z$ is neither $x$ nor $y$. Thus, using (17) and the particular structure of the function $f$, we derive that

$$\begin{aligned}\Phi_{(\phi;\, y)}(\omega, \gamma_1, r) &= f[(\phi;\, y)_r^{\omega,\gamma}] - f[(\phi^{xy};\, y)_r^{\omega,\gamma}]\\ &= I_D(\{y\}_r^{\omega}) - I_D(\{x\}_r^{\omega}) \qquad (19)\end{aligned}$$

for any $\omega$ independently of what particular $\gamma_1$ we have chosen.

Next, we recall that the marginal motion in the stirring system of a particle which originates from any $z \in \mathbb{Z}^d$ coincides in law with the simple symmetric random walk which starts from $z$ (recall that this random walk is denoted by $Z_t^z$, $t \geqslant 0$). Thus,

$$\begin{aligned}\mathbb{E}_{\Omega}[I_D(\{y\}_r^{\omega}) - I_D(\{x\}_r^{\omega})] &= \mathbb{P}[Z_r^y \in D] - \mathbb{P}[Z_r^x \in D]\\ &= \mathbb{P}[Z_r^y \in +\partial_1 D] - \mathbb{P}[Z_r^y \in -\partial_1 D - e_1] \quad (20)\end{aligned}$$

where

$$+\partial_1 D := \{z \in \Delta_1 D\colon z^1 = h\}, \qquad -\partial_1 D - e_1 := \{z \in \Delta_1 D\colon z^1 = -h-1\}$$

The last equality in (20) is obtained by coupling the random walks $Z_\cdot^x$ and $Z_\cdot^y$ in such a way that they move simultaneously in the same direction. Using the same coupling, one also gets the following implication:

$$\gamma_1 \in \Gamma \text{ is such that } y_r^{\gamma_1} \in D \text{ and } x_r^{\gamma_1} \notin D \Leftrightarrow \gamma_1 \in \Gamma \text{ is such that } y_r^{\gamma_1} \in +\partial_1 D$$

Thus

$$\mu_\Gamma[\gamma_1 \in \Gamma \text{ is such that } y_r^{\gamma_1} \in D \text{ and } x_r^{\gamma_1} \notin D] = \mathbb{P}[Z_r^y \in +\partial_1 D] \quad (21)$$

Combining (19)–(21), we conclude that

$$\begin{aligned}&\mathbb{E}_{\Omega\times\Gamma}[\Phi_{(\phi;\, y)}(\omega, \gamma, r) I_{\{y_r^\gamma \in D,\, x_r^\gamma \notin D\}}]\\ &\quad = \mathbb{P}[Z_r^y \in +\partial_1 D](\mathbb{P}[Z_r^y \in +\partial_1 D] - \mathbb{P}[Z_r^y \in -\partial_1 D - e_1]) \quad (22)\end{aligned}$$

The same way of reasoning gives that

$$\begin{aligned}&\mathbb{E}_{\Omega\times\Gamma}[\Phi_{(\phi;\, y)}(\omega, \gamma, r) I_{\{y_r^\gamma \notin D,\, x_r^\gamma \in D\}}]\\ &\quad = \mathbb{P}[Z_r^y \in -\partial_1 D - e_1](\mathbb{P}[Z_r^y \in -\partial_1 D - e_1]\\ &\qquad - \mathbb{P}[Z_r^y \in +\partial_1 D]) \qquad (23)\end{aligned}$$

Next, because of the particular form of the function $f$, we have that $\Phi(\omega, \gamma, r)=0$ for every $\omega \in \Omega^A$ when $\gamma \in \Gamma$ is such that both $x_r^\gamma$ and $y_r^\gamma$ either belong or do not belong to $D$. Consequently,

$$\mathbb{E}_{\Omega \times \Gamma}[\Phi_{(\phi;\, y)}(\omega, \gamma, r) I_{\{(y_r^\gamma \in D \text{ and } x_r^\gamma \in D) \text{ or } (y_r^\gamma \notin D \text{ and } x_r^\gamma \notin D)\}}]=0 \tag{24}$$

It is important to observe that (22)–(24) hold for any $(\phi;\, y) \in X^*$ such that $\phi(x)=0$ and do not depend on the values of $\phi$ on $\mathbb{Z}^d \setminus \{x, y\}$. If, to the contrary, $\phi(x)=1$, then $\phi^{xy}=\phi$, which yields that $\Phi_{(\phi;\, y)}(\omega, \gamma, r) \equiv 0$. Combining the latter with (22)–(24), we finally conclude that

$$\begin{aligned} 0 &\leqslant \mathbb{E}_{\Omega \times \Gamma} \Phi_{(\phi;\, y)}(\omega, \gamma, r) \\ &\leqslant (\mathbb{P}[Z_r^y \in +\partial_1 D]-\mathbb{P}[Z_r^y \in -\partial_1 D-e_1])^2 \\ &\leqslant (\mathbb{P}[Z_r^y \in \Delta_1 D])^2 \end{aligned} \tag{25}$$

for any $(\phi;\, y) \in X^*$ independently of the values of $\phi$ on $\mathbb{Z}^d \setminus \{y\}$.

We now start the reasoning which will connect (25) to the quantity we wish to estimate.

From (14) and (13) we have that

$$\begin{aligned} ([V-U]\, g)(\phi;\, y) = \sum_{x:\, x \sim y} \frac{1}{2d} [&g((\phi;\, y))-g((\phi^{xy};\, y)) \\ &+g((\phi^{xy};\, x))-g((\phi;\, x))] \end{aligned} \tag{26}$$

for any $(\phi;\, y) \in X^*$. In what follows $g$ from (26) will be taken equal to $U(r) f$. Since $f$ is a cylinder function, then (by the Hille–Yosida theorem, for example[6]) $U(r) f$ is in the domain of definition of $U$. It may be checked directly that it is also in the domain of the definition of $V$. The fact that $U(r) f$ is defined on the whole $\hat{X}$ should not be confusing since $(\phi;\, v) \in X^*$.

It is convenient to introduce an auxiliary family of operators $\{F_j^+, F_j^-, j=1, \ldots, d\}$ acting on the set of the functions from $\hat{X}$ to $\mathbb{R}$ by [below $\phi(u, v) \equiv \phi^{uv}$ to avoid complicated superscripts]

$$\begin{aligned} F_j^{\pm}\, g(\phi;\, y) := \frac{1}{2d} [&g((\phi;\, y))-g((\phi(y \pm e_j, y);\, y)) \\ &+g((\phi(y \pm e_j, y);\, y \pm e_j))-g((\phi;\, y \pm e_j))] \end{aligned}$$

for $(\phi; y) \in \hat{X}$. Then, using the integration by parts formula, (26), and the above-defined operators, we have that

$$\begin{aligned}(V(s) - U(s)f)(\chi; v) &= \left(\int_0^s V(s-r)[V-U]\, U(r) f\, dr\right)(\chi; v)\\ &= \sum_{j=1}^{d} \left(\int_0^s V(s-r)[F_j^+ + F_j^-]\, U(r) f\, dr\right)(\chi; v) \qquad (27)\end{aligned}$$

It follows straightforwardly from the corresponding definitions that

$$F_1^+ U(r) f(\phi; y) = \frac{1}{2d} \mathbb{E}_{\Omega \times \Gamma} \Phi_{(\phi; y)}(\omega, \gamma, r), \qquad \forall (\phi; y) \in X^* \qquad (28)$$

From the above identity and (25) we have that for any $(\chi; v) \in X^*$,

$$\begin{aligned}&V(s-r)\, F_1^+ U(r) f(\chi; v)\\ &\quad = \mathbb{E}^{(\chi; v)}[F_1^+ U(r) f(\phi; y) \mid (\chi; v)_{s-r} = (\phi; y)]\\ &\quad \leqslant \frac{1}{2d} \mathbb{E}^{(\chi; v)}[\{\mathbb{P}[Z_r^{v_{s-r}} \in \Delta_1 D]\}^2]\\ &\quad = \frac{1}{2d} \sum_{y \in \mathbb{Z}^d} \mathbb{P}[Z_{s-r}^v = y]\{\mathbb{P}[Z_r^y \in \Delta_1 D]\}^2 \qquad (29)\end{aligned}$$

where $\mathbb{E}^{(\chi; v)}$ denotes the mathematical expectation with respect to the law of the process $V$ which started from $(\chi; v)$; $(\chi; v)_{s-r}$ and $v_{s-r}$ are, respectively, the state of this process and the position of the marked particle in this process at time $s-r$. Observe that the last equality was obtained using the fact that the marginal motion of this particle coincides in law with a simple symmetric random walk in $\mathbb{Z}^d$ starting from $v$. Also observe that $(\chi; v)_t \in X^*$ for all $t \geqslant 0$, which made it possible to apply (28) in deriving the intermediate inequality in (29).

Reasoning as for (29) but using another inequality in (25), we also have that

$$0 \leqslant V(s-r)\, F_1^+ U(r) f(\chi; v) \qquad (30)$$

Observe now that changing appropriately the definition of $\Phi_{(\phi; y)}$, one derives that (29), (30) hold for any $F_j^{\pm}$ with $\Delta_1 D$ being changed respectively to $\Delta_j D$. Applying (27), one then derives the assertion of the lemma. ▌

We will now show how the processes $U$ and $V$ relate to the process $\eta_{\cdot}$. Let $\chi$ be a configuration of $A$ and $B$ particles on $\mathbb{Z}^d$ such that an arbitrarily fixed site, say $v$, is occupied by both $A$ and $B$ particles. Let ${}^A\chi$ denote the configuration obtained from $\chi$ by erasing from it all its $B$ particles. Recall the definitions given after the proof of Proposition 1 and consider the configuration $({}^A\chi; v) \in X^*$. We recall that $({}^A\chi; v)$ means the configuration which has an unmarked particle at each site where $\chi$ has an $A$ particle, and it has a marked particle at $v$. Then, from the definition of the processes $\eta_\bullet$, $V$, and $U$, we derive that the pair $(\bigcup_{y:\chi(y)=A} {}^A\eta_t^y; {}^A\eta_t^v)$ [respectively, $(\bigcup_{y:\chi(y)=A} {}^A\eta_t^y; {}^B\eta_t^v)$] is distributed exactly like the configuration of all unmarked particles and one marked particle in the process $U$ (respectively, $V$) at time $t$ given the initial configuration of this process was $({}^A\chi; v)$. This fact leads to the following [$f$ below is from (15)]:

$$\begin{aligned}
&\mathbb{E}_\chi\left[I_D({}^A\eta_t^v) \sum_{y:\chi(y)=A} I_D({}^A\eta_t^y)\right] = V(t)\, f({}^A\chi; v) \\
&\mathbb{E}_\chi\left[I_D({}^B\eta_t^v) \sum_{y:\chi(y)=A} I_D({}^A\eta_t^y)\right] = U(t)\, f({}^A\chi; v)
\end{aligned} \tag{31}$$

Let us postulate that if in a configuration, two alive particles occupy the same site, then they change their label to "dead" immediately after the process $\eta_\bullet$ starts from this configuration. Thus, if $\eta_\bullet$ starts from $\chi$, then $k(v) = v$ independently of the evolution of $\eta_\bullet$. Hence, (31) lead to

$$\begin{aligned}
&\mathbb{E}_\chi\left[\{I_D({}^A\eta_t^v) - I_D({}^B\eta_t^{k(v)})\} \sum_{y:\chi(y)=A} I_D({}^A\eta_t^y)\right] \\
&\quad = [V(t) - U(t)]\, f({}^A\chi; v)
\end{aligned} \tag{32}$$

The left-hand side of (32) is exactly the quantity we wish to estimate in the following Lemma 2. Thus, (32) suggests that we use Lemma 1 for this need. However, the assumption $k(v) \equiv v$ which allowed us to derive (32) is never true in the case when $\eta_0$ contains at most one particle per site. In fact, given $v \in \mathbb{Z}^d$, we know nothing about the distribution of $k(v)$ in the process $\eta_\bullet$. In the following lemma, we overcome these difficulties and estimate the left-hand side of (32).

**Lemma 2.** Under the assumptions and designations of Lemma 1, for any $\eta_0 \in \mathscr{Y}$ and any $x \in \mathbb{Z}^d$ such that $\eta_0(x) = A$ and all $t \geq 0$, it holds that

$$\begin{aligned}
&\mathbb{E}_{\eta_0}[(I_D(\eta_t^x) - I_D(\eta_t^{k(x)})) \sum_{y:\eta_0(y)=A} I_D(\eta_t^y)] \\
&\quad \leq \frac{1}{d} \sum_{i=1}^{d} \int_0^t \sum_{y \in \mathbb{Z}^d} \mathbb{P}[Z_{t-r}^x = y]\{\mathbb{P}[Z_r^y \in \Delta_i D]\}^2\, dr
\end{aligned} \tag{33}$$

*Proof.* Fix $t>0$ and let $\eta_0$ and $x$ satisfy the lemma's assumptions. Choose arbitrarily $v\in\mathbb{Z}^d$, $\eta\in\mathscr{X}$ such that $\eta(x)=A\cup B$ and $\tau\leqslant t$. Reasoning as for (31)–(32), we obtain that

$$\mathbb{E}_{\eta_0}\left[(I_D(\eta_t^x)-I_D(\eta_t^{k(x)}))\sum_{y:\eta_0(y)=A} I_D(\eta_t^y)\,\middle|\,\eta_\tau=\eta,\ \tau(x)=\tau,\ \eta_\tau^x=v\right]$$

$$=\mathbb{E}_{\eta_0}[(V(t-\tau)-U(t-\tau))\,f({}^A\eta;v)\mid\eta_\tau=\eta,\ \tau(x)=\tau,\ \eta_\tau^x=v] \quad (34)$$

where ${}^A\eta$ is obtained from $\eta$ by erasing all its $B$ particles. Now multiply both sides of (34) by $\mathbb{P}_{\eta_0}[\eta_\tau=\eta,\ \tau(x)=\tau,\ \eta_\tau^x=v]$ and take a sum over all $\eta\in\mathscr{X}$, $v\in\mathbb{Z}^d$, $\tau\in[0,t]$. After this procedure, the left-hand side of (34) equals the left-hand side of (33), while the right-hand side is ($l$ below is the Lebesgue measure on $\mathbb{R}$)

$$\leqslant\sum_{j=1}^d\frac{1}{d}\sum_{v\in\mathbb{Z}^d}\int_0^t l(d\tau)\,\mathbb{P}_{\eta_0}[\tau(x)=\tau,\ \eta_\tau^x=v]$$

$$\times\mathbb{E}_{\eta_0}\left[\int_0^{t-\tau}\sum_{y\in\mathbb{Z}^d}\mathbb{P}[Z_{t-\tau-r}^v=y]\right.$$

$$\left.\times\{\mathbb{P}[Z_r^y\in\Delta_jD]\}^2\,dr\,\middle|\,\tau(x)=\tau,\ \eta_\tau^x=v\right] \quad (35)$$

where in deriving (35) we used Lemma 1 and the fact that the r.h.s. of (16) does not depend on the values of $\eta$ on $\mathbb{Z}^d\backslash\{v\}$.

Let $\Theta$ denote the set of all paths of a simple symmetric random walk in $\mathbb{Z}^d$ which starts from $x$. An element of $\Theta$ will be denoted by $\theta$; then, $\theta_q$ is the position of the path $\theta$ at time $q$ and $\theta_{[a,b]}$ is the portion of this path from time $a$ to time $b$. For $\eta_0$ and $x$ fixed above and $\theta\in\Theta$, let $\mathscr{P}_\theta$ be the distribution of $\tau(x)$ given that the particle which started from $x$ in $\eta_0$ followed the path $\theta$ from 0 to $\infty$, i.e.,

$$\mathscr{P}_\theta([a,b]):=\mathbb{P}_{\eta_0}[\tau(x)\in[a,b]\mid\eta_{[0,\infty]}^x=\theta_{[0,\infty]}]$$

$$=\mathbb{P}_{\eta_0}[\tau(x)\in[a,b]\mid\eta_{[0,b]}^x=\theta_{[0,b]}],\qquad\forall 0\leqslant a<b\leqslant\infty \quad (36)$$

where the last equality holds because the motion (in the process $\eta_\cdot$) of a particle after it has changed its label to "dead" does not depend on how this particle arrived at the point of $\mathbb{Z}^d$ where it was annihilated.

Using (36) and the fact that the marginal motion of a particle is a simple symmetric random walk, we write

$$\sum_{v \in \mathbb{Z}^d} \int_0^t l(d\tau)\, \mathbb{P}_{\eta_0}[\tau(x) = \tau,\ \eta_\tau^x = v]$$

$$= \sum_{\theta \in \Theta} \int_0^t \mathscr{P}_\theta(d\tau)\, \mathbb{P}_{\eta_0}[\eta^x_{[0,\tau]} = \theta_{[0,\tau]}]$$

$$= \sum_{\theta \in \Theta} \int_0^t \mathscr{P}_\theta(d\tau)\, \mathbb{P}[Z^x_{[0,\tau]} = \theta_{[0,\tau]}] \tag{37}$$

Plug (37) into (35) now and change appropriately $Z^v_{t-\tau-r}$ to $Z^{\theta_\tau}_{t-\tau-r}$. After this procedure, the $j$th term of (35) becomes

$$d^{-1} \sum_{\theta \in \Theta} \int_0^t \mathscr{P}_\theta(d\tau)\, \mathbb{P}[Z^x_{[0,\tau]} = \theta_{[0,\tau]}]$$

$$\times \int_0^{t-\tau} \sum_{y \in \mathbb{Z}^d} \mathbb{P}[Z^{\theta_\tau}_{t-\tau-r} = y]\{\mathbb{P}[Z^y_r \in \Delta_j D]\}^2\, dr$$

$$= d^{-1} \sum_{\theta \in \Theta : \theta_t \in \Delta_j D} \int_0^t \mathscr{P}_\theta(d\tau) \int_0^{t-\tau} \mathbb{P}[Z^x_{[0,t-r]} = \theta_{[0,t-r]}]$$

$$\times \{\mathbb{P}[Z^{\theta_{t-r}}_{[t-r,t]} = \theta_{[t-r,t]}]\}^2\, dr$$

$$\leqslant d^{-1} \sum_{\theta \in \Theta : \theta_t \in \Delta_j D} \int_0^t \mathbb{P}[Z^x_{[0,t-r]} = \theta_{[0,t-r]}]$$

$$\times \{\mathbb{P}[Z^{\theta_{t-r}}_{[t-r,t]} = \theta_{[t-r,t]}]\}^2\, dr$$

$$= d^{-1} \int_0^t \sum_{y \in \mathbb{Z}^d} \mathbb{P}[Z^x_{t-r} = y]\{\mathbb{P}[Z^y_r \in \Delta_j D]\}^2\, dr \tag{38}$$

Summing the r.h.s. of (38) over all $j = 1,\ldots, d$ gives the r.h.s. of (33). ▮

**Lemma 3.** Under the assumptions and designations of Lemma 1, it holds that

$$\mathbb{E}\left[\sum_{x : \eta_0(x) = A} (I_D(\eta^x_t) - I_D(\eta^{k(x)}_t)) \sum_{y : \eta_0(y) = A} I_D(\eta^y_t)\right]$$

$$\leqslant C\sqrt{t}\, |\Delta_1 D| \qquad \text{for all} \quad t \geqslant 0 \tag{39}$$

*Proof.* If $W_s^z$, $s \geqslant 0$, denotes a one-dimensional random walk starting from $z \in \mathbb{Z}$, then for any $y = (y^1, ..., y^d) \in \mathbb{Z}^d$ and any $j = 1, ..., d$,

$$\mathbb{P}[Z_r^y \in \Delta_j D] \leqslant \mathbb{P}[W_r^{y^j} \in \{-h-1, -h, h, h+1\}] \leqslant \frac{4C_2}{r^{1/2}} \qquad \forall r \geqslant 0 \quad (40)$$

where the last inequality holds because

$$\sup_{z \in \mathbb{Z}} \mathbb{P}[W_r^u = z] \leqslant \frac{C_2}{r^{1/2}} \qquad \text{for an appropriate } C_2 > 0 \text{ and } \forall u \in \mathbb{Z}, \quad \forall r \geqslant 0$$

Substituting (40) in (33), we have that for any $\eta_0$,

$$\begin{aligned}\sum_{x : \eta_0(x) = A} & \mathbb{E}_{\eta_0}\left[(I_D(\eta_t^x) - I_D(\eta_t^{k(x)})) \sum_{y : \eta_0(y) = A} I_D(\eta_t^y)\right] \\ &\leqslant \frac{1}{d} \sum_{j=1}^d \sum_{x : \eta_0(x) = A} \mathbb{P}[Z_t^x \in \Delta_j D] \int_0^t \frac{4C_2}{r^{1/2}}\, dr \\ &\leqslant \frac{1}{d} \sum_{j=1}^d C_3 \sqrt{t}\, |\Delta_j D| = C\sqrt{t}\, |\Delta_1 D|\end{aligned}$$

where we used that $\sum_{x \in \mathbb{Z}^d} \mathbb{P}[Z_t^x \in G] = |G|$ for any finite $G \in \mathbb{Z}^d$. Integrating the above inequality over all $\eta_0$ with respect to the initial measure of the process $\eta_\cdot$, we get the assertion of the lemma. ∎

## 4. UPPER BOUND

This section will provide an asymptotic upper bound for $\rho(t)$.

**Lemma 4.** Under the assumptions and designations of Lemma 1, it holds that

$$\begin{aligned}\mathbb{E}&\left[\sum_{x : \eta_0(x) = A} (I_D(\eta_t^{k(x)}) - I_D(\eta_t^x)) \sum_{\substack{y : \eta_0(y) = A \\ y \neq x}} I_D(\eta_t^y)\right] \\ &\leqslant C\sqrt{t}\, |\Delta_1 D| \qquad \text{for all} \quad t \geqslant 0\end{aligned} \qquad (41)$$

*Proof.* In the proof of Lemma 2.3 of ref. 2, estimate the right-hand side of (2.21) from above by $(\mathbb{P}[Z_r^y \in \Delta_1 D])^2/2d$. Continuing then as in the proof of that lemma, we substitute (2.22) from ref. 2 by ($F_j^+$ from ref. 2 is actually $-F_j^+$, which was defined in the proof of Lemma 1)

$$V(s-r)\, F_1^+\, U(r)\, f(\chi; v) \leqslant \frac{1}{d} \sum_{y \in \mathbb{Z}^d} \mathbb{P}[Z_{t-r}^x = y]\{\mathbb{P}[Z_r^y \in \Delta_1 D]\}^2$$

Then Lemma 2.3 of ref. 2 will assert that

$$(U(s)-V(s))\,f(\chi;v)$$
$$\leqslant \frac{1}{d}\sum_{j=1}^{d}\int_0^s \sum_{y\in\mathbb{Z}^d} \mathbb{P}[Z_{t-r}^x=y]\{\mathbb{P}[Z_r^y\in \Delta_j D]\}^2\,dr \tag{42}$$

The same argument as we used in the proof of Lemma 2 of Section 3 shows then that for each $\eta_0$ and $x$ such that $\eta_0(x)=A$,

$$\mathbb{E}_{\eta_0}\left[(I_D(\eta_t^{k(x)})-I_D(\eta_t^x))\sum_{\substack{y:\eta_0(y)=A\\ y\neq x}} I_D(\eta_t^y)\right]$$
$$\leqslant \text{r.h.s. of (42) with } s=t \tag{43}$$

The above inequality then yields (41) by the same reasoning as used in the proof of Lemma 3 of Section 3. ∎

Take now $R_t:=\delta t^{1/2}$, where the value of $\delta$ will be chosen later independently of $t$. Let $R_t$ be the side of the cube $D$. Then

$$C\sqrt{t}\,|\Delta_1 D|=4C\sqrt{t}\,(R_t)^{d-1}=C_4R_t^d \text{ for an appropriate } C_4 \tag{44}$$

Using (41) and (44) to estimate the last mathematical expectation in (2.39) from the proof of Lemma 2.6 in ref. 2, we then obtain the following strengthened version of the latter:

**Lemma 5.** Let $D$ be the cube in $\mathbb{Z}^d$ with side $R_t$ centered at the origin. Then for all sufficiently large $t$,

$$\mathbb{E}[|\mathscr{D}_R(s)|]\left(:=\mathbb{E}\left[\left|\sum_{x:\eta_0(x)=A} I_D(\xi_s^x)-\sum_{y:\eta_0(y)=B} I_D(\xi_s^y)\right|\right]\right)$$
$$\leqslant C_6R_t^{d/2} \qquad \text{for all} \quad s\in[t/2,t]$$

Now, in the proof of Theorem 1.1 of ref. 2 substitute the definition of the function $g$, (2.47), by

$$g(t):=\begin{cases} C_5t^{-d/4} & \text{when} \quad d\leqslant 4\\ C_5t^{-1} & \text{when} \quad d>4\end{cases}$$

then change respectively (2.48) and use the above Lemma 5 instead of Lemma 2.6 from ref. 2 throughout the whole proof of Theorem 1.1 of ref. 2.

(We recall that the values of the absolute constants $C_5$ and $\delta$ are determined in the course of this proof.) This will strengthen Theorem 1.1 of ref. 2 in the following manner.

**Proposition 2.** There exists a finite constant $C(d)$ such that when $t$ is large enough

$$\begin{aligned} \rho(t) &\leqslant C(d)\, t^{-d/4} \qquad \text{when} \quad d \leqslant 4 \\ &\leqslant C(d)\, t^{-1} \qquad \text{when} \quad d > 4 \end{aligned}$$

## 5. LOWER BOUND FOR DIMENSIONS $d \geqslant 4$

In this section we show that $\rho(t)$ is bounded from below by $\text{const} \times t^{-1}$. Together with Proposition 2 it shows that $t^{-1}$ is the correct exponent of the decay of the density in the AE in the dimensions $d \geqslant 4$. The idea of the reasoning that establishes $\rho(t) \geqslant \text{const} \times t^{-1}$ is borrowed from the ref. 3, Section 3.

The concepts used below are defined exclusively either in the course of this section or in Section 2, though the notations may coincide with what we used in Sections 3 and 4.

For a configuration $\gamma \in \mathscr{Y} := \{A, B, 0\}^{\mathbb{Z}^d}$ of $A$ and $B$ particles on $\mathbb{Z}^d$ and $x_1 \in \mathbb{Z}^d$ such that $\gamma(x_1) = A$, we define the process $X_t^{x_1}$, $t \geqslant 0$, in the following manner: its state space is $\mathbb{Z}^d$ and its path is a function of $\omega$ constructed by the following rule: for each $\omega \in \Omega$, consider $\eta_t[\omega]$, $t \geqslant 0$, with $\eta_0 = \gamma$; then for $s \in [0, \tau(x_1)[\gamma, \omega]]$, $X_s^{x_1}[\gamma, \omega]$ coincides with the path in this $\eta_{\boldsymbol{\cdot}}[\omega]$ of the alive $A$ particle which started from $x_1$; from $\tau(x_1)$ on, it coincides with the path (in the same $\eta_{\boldsymbol{\cdot}}[\omega]$) of the dead $B$ particle which annihilated that $A$ particle; this happens until the first moment when this $B$ particle meets an alive $A$ particle; from the meeting time until the $A$ particle which has been met is alive (in $\eta_{\boldsymbol{\cdot}}[\omega]$), the path of this particle coincides with the path of $X_{\boldsymbol{\cdot}}^{x_1}[\gamma, \omega]$. We note that in order to indicate explicitly the dependence of the process of the configuration $\gamma$, we put $\gamma$ in the square brackets because it will be random in the sequel (recall Remark on Notation in Section 2). Formally, $X_t^{x_1}$, $t \geqslant 0$, can be written as

$$\begin{aligned} X_s^{x_1} &= {}^A\eta_s^{x_k} \qquad \text{for} \quad s \in [r_k, \tau(x_k)) \\ &= {}^B\eta_s^{y_k} \qquad \text{for} \quad s \in [\tau(x_k), r_{k+1}) \end{aligned}$$

where the sequence

$$\{x_k = x_k[\gamma, \omega],\ y_k = y_k[\gamma, \omega],\ r_k = r_k[\gamma, \omega]\}_{k \in \mathbb{Z}^+}$$

is defined recursively in the following way: $y_k$ is the initial position of the annihilating companion of the $A$ particle which started from $x_k$; $r_1 \equiv 0$ and for $k = 1, 2, \ldots,$

$$\begin{aligned} r_{k+1} := \inf\{t > r_k \colon {}^B\eta_t^{y_k}[\omega] = {}^A\eta_t^{x}[\omega] \\ \text{for some } x \text{ s.t. } \gamma(x) = A \text{ and } \tau(x)[\gamma, \omega] > t\} \end{aligned}$$

Finally, $x_{k+1}$, $k \geqslant 1$, is such that

$$ {}^B\eta_{r_{k+1}}^{y_k}[\omega] = {}^A\eta_{r_{k+1}}^{x_{k+1}}[\omega] $$

Next, for a configuration $\gamma \in \mathscr{Y}$ of $A$ and $B$ particles on $\mathbb{Z}^d$ and $x_1 \in \mathbb{Z}^d$ such that $\gamma(x_1) = 0$, we define the process $\bar{X}_t^{x_1}$, $t \geqslant 0$, in the following manner: We first introduce the configuration $\gamma^{\cdot y} \in \mathscr{Y}$ by

$$\gamma^{\cdot y}(z) = \begin{cases} \gamma(z) & \text{if } z \neq y \\ A & \text{if } z = y \end{cases} \qquad \text{for} \quad \gamma \in \{A, B, 0\}^{\mathbb{Z}^d} \text{ and } y \in \mathbb{Z}^d \tag{45}$$

and then postulate that

$$\bar{X}_s^{x_1}[\gamma, \omega] = X_s^{x_1}[\gamma^{\cdot y}, \omega] \qquad \text{for all } s \geqslant 0 \text{ and all } \omega \in \Omega$$

Directly from the construction of the processes $X$ and $\bar{X}$ we derive the following result:

**Lemma 6.** For a configuration $\alpha \in \mathscr{Y}$, let $\xi_t(x)[\alpha, \omega]$ denote the value that the annihilating exclusion process attains at the site $x \in \mathbb{Z}^d$ at time $t$ on the percolation substructure $\omega \in \Omega$ given $\xi_0 = \alpha$. Let $\gamma \in \mathscr{Y}$ and $y \in \mathbb{Z}^d$ be such that $\gamma(0) = A$. Then for all $t \geqslant 0$:

(i) $\xi_t(z)[\gamma, \omega] = \xi_t(z)[\gamma^{\cdot y}, \omega]$ for any $z \neq X_t^y[\gamma^{\cdot y}, \omega] = \bar{X}_t^y[\gamma, \omega]$

(ii) $\xi_t(X_t^y[\gamma^{\cdot y}, \omega])[\gamma^{\cdot y}, \omega]$ is either $A$ or 0, while $\xi_t(\bar{X}_t^y[\gamma, \omega])[\gamma, \omega]$ is either $B$ or 0

(iii) $\xi_t(X_t^y[\gamma^{\cdot y}, \omega])[\gamma^{\cdot y}, \omega] = A$ (resp. 0) if and only if $\xi_t(\bar{X}_t^y[\gamma, \omega])[\gamma, \omega] = 0$ (resp., $B$).

Fix $t \geqslant 1$. Let $\Gamma$ be an abstract probability space on which two families of random variables $\{\phi_x\}_{x \in \mathbb{Z}^d}$ and $\{\delta_x\}_{x \in \mathbb{Z}^d}$ are defined in such a way that they are independent (within each family, between the families, and of everything else), identically distributed within the same family, and for each $x \in \mathbb{Z}^d$,

$$\mathbb{P}[\phi_x = A] = \mathbb{P}[\phi_x = B] = \rho$$

$$\mathbb{P}[\phi_x = 0] = 1 - 2\rho$$

$$\mathbb{P}[\delta_x = 1] = 1 - \mathbb{P}[\delta_x = 0] = (4t)^{-1}$$

To each $\gamma \in \Gamma$ there corresponds a configuration from $\mathscr{Y}$ in a natural way: a site $x \in \mathbb{Z}^d$ is occupied by an $A$ or $B$ particle or is empty in this configuration if and only if the value of $\phi_x[\gamma]$ is, respectively, $A$, $B$, or 0. The configuration that corresponds to $\gamma \in \Gamma$ will be denoted by the same symbol $\gamma$. (This should not yield confusion, since when $\gamma$ appears as an argument of a variable it will always be clear from the context and from the definition of this variable whether $\gamma$ is an element from $\Gamma$ or the corresponding element from $\mathscr{Y}$.) One easily sees that $\{\phi_x\}_{x \in \mathbb{Z}^d}$ is constructed in such a way that the distribution of particles in $\gamma$ coincides with the initial distribution of particles in the annihilating exclusion process. The family $\{\delta_x\}_{x \in \mathbb{Z}^d}$ is designed to construct a set $\mathscr{A}_t$ of sites which is sufficiently "thin" in $\mathbb{Z}^d$. Namely, for each $\gamma \in \Gamma$, we define $\mathscr{A}_t = \mathscr{A}_t[\gamma] := \{x \in \mathbb{Z}^d\colon\ \delta_x = 1$ and $\phi_x \neq B\}$.

Let $Y^x_\cdot[\gamma, \cdot]$ stand for $X^x_\cdot[\gamma, \cdot]$ if $\gamma(x) = A$ and stand for $\bar{X}^x_\cdot[\gamma, \cdot]$ if $\gamma(x) = 0$. The following lemma concerns a particular property of the set of the processes $Y$ originated from the points of $\mathscr{A}_t$. To formulate this lemma, we introduce more notations: For distinct $x$ and $y$ from $\mathscr{A}_t$, the sign $Y^x_s \leftrightarrow Y^y_s$ will mean that at time $s$ either $Y^x_\cdot$ and $Y^y_\cdot$ interchange their positions (i.e., $Y^x_s = Y^y_{s-}$, $Y^y_s = Y^x_{s-}$, and $Y^x_s \neq Y^x_{s-}$), or one of them jumps to the site which is occupied by another (i.e., $Y^x_s = Y^y_s$ and $Y^x_{s-} \neq Y^y_{s-}$). The sign $\mathscr{B}_y = \mathscr{B}_y[\gamma, \omega]$ will be a shorthand for the following expression:

$$Y^y_t[\gamma, \omega] = 0 \text{ and } Y^y_s[\gamma, \omega] \not\leftrightarrow Y^z_s[\gamma, \omega]$$
$$\text{for all } s \in [0, t] \text{ and all } z \in \mathscr{A}_t[\gamma] \setminus \{y\}$$

**Lemma 7.** $\mathbb{P}[\exists! y \in \mathscr{A}_t\colon \mathscr{B}_y] \geqslant (1-\rho)^2/(8t)$, for all sufficiently large $t$.

*Proof.* Since for distinct $y_1$, $y_2$,

$$\{\omega, \gamma\colon y_1 \in \mathscr{A}_t \text{ and } \mathscr{B}_{y_1}\} \cap \{\omega, \gamma\colon y_2 \in \mathscr{A}_t \text{ and } \mathscr{B}_{y_2}\} = \varnothing$$

then the probability of interest equals $\sum_{y \in \mathbb{Z}^d} \mathbb{P}[y \in \mathscr{A}_t$ and $\mathscr{B}_y]$. Using then the invariance of the measures on $\Gamma$ and on $\Omega$ with respect of the translations of $\mathbb{Z}^d$, we rewrite this sum in the following equivalent form:

$$\sum_{y \in \mathbb{Z}^d} \mathbb{P}[0 \in \mathscr{A}_t,\ Y^0_t = -y \text{ and } Y^0_s \not\leftrightarrow Y^{z-y}_s$$
$$\text{for all } s \in [0, t] \text{ and all } z - y \in \mathscr{A}_t \setminus \{0\}]$$
$$= \mathbb{P}[0 \in \mathscr{A}_t,\ Y^0_s \not\leftrightarrow Y^z_s \text{ for all } s \in [0, t] \text{ and all } z \in \mathscr{A}_t \setminus \{0\}]$$

To prove that the last probability is at least $(1-\rho)^2/(8t)$, we reason in the following manner.

Assume 0 and some $x$ belong to $\mathscr{A}_t$. Due to the construction of $X$ and $\bar{X}$, we have that $Y^0_\cdot$ and $Y^x_\cdot$ interact by the rules of the stirring mechanism when the evolution of both is ruled by $\omega^A$ or by $\omega^B$, whereas they behave like two independent random walks when the evolution of one of them is ruled by $\omega^A$ and that of the other by $\omega^B$. Using this fact, we derive that

$$\mathbb{P}[Y^0_s \leftrightarrow Y^x_s \text{ for some } s \leqslant t \mid 0, x \in \mathscr{A}_t] \leqslant \mathbb{P}[{}^2Z^0_s = x \text{ for some } s \leqslant t]$$

where ${}^2Z^0_s$, $s \geqslant 0$, is a rate-2 simple symmetric random walk in $\mathbb{Z}^d$ starting from 0. Thus,

$$\begin{aligned} &\mathbb{P}[Y^0_s \leftrightarrow Y^x_s \text{ for some } x \in \mathscr{A}_t \backslash \{0\} \text{ and some } s \leqslant t \mid 0 \in \mathscr{A}_t] \\ &\qquad \leqslant \mathbb{E}[\#\{x\colon {}^2Z^0_s = x \text{ for some } s \leqslant t \text{ and } x \in \mathscr{A}_t \backslash \{0\}\}] \end{aligned} \tag{46}$$

Given that $x \neq 0$ belongs to a path ${}^2Z^0_s$, $s \in [0, t]$, the probability that it belongs to $\mathscr{A}_t$ equals $(1-\rho)/(4t)$ (by the definition of $\{\phi_x\}$ and $\{\delta_x\}$). The mean number of distinct sites of $\mathbb{Z}^d$ visited up to time $t$ by a rate-2 simple symmetric random walk is not greater than $2t$, the mean number of the jumps of this walk. Consequently, (46) is not greater than $(1-\rho)/2$, which yields that

$$\mathbb{P}[Y^0_s \nleftrightarrow Y^z_s \text{ for all } s \in [0, t] \text{ and all } z \in \mathscr{A}_t \backslash \{0\} \mid 0 \in \mathscr{A}_t] \geqslant (1-\rho)/2$$

Since $\mathbb{P}[0 \in \mathscr{A}_t] = (1-\rho)/(4t)$, the assertion of the lemma follows. ∎

**Proposition 3.** There is a positive constant $c^*$ such that for sufficiently large $t$,

$$c^* t^{-1} \leqslant \rho(t)$$

*Proof.* Pick $\omega \in \Omega$, $y \in \mathbb{Z}^d$, and $\gamma \in \Gamma$ for which

$$\begin{aligned} &Y^y_t[\gamma, \omega] = 0 \text{ and } Y^y_s[\gamma, \omega] \nleftrightarrow Y^z_s[\gamma, \omega] \\ &\qquad \text{for all } s \in [0, t] \text{ and all } z \in \mathscr{A}_t[\gamma] \backslash \{y\} \end{aligned} \tag{47}$$

Assume for concreteness $\gamma(y) = 0$ (the same reasoning will pertain to the opposite case), so that $Y^y_\cdot$ in (47) stands for $\bar{X}^y_\cdot$. Consider now the configuration $\gamma^y$ [recall the definition (45)]. By the construction of the paths $X$ and $\bar{X}$, we have that $\bar{X}^y_s[\gamma, \omega] = X^y_s[\gamma^y, \omega]$ for all $s \in [0, t]$. By our choice of $\gamma$, $\omega$, and $y$, $\bar{X}^y_s[\gamma, \omega]$ does not intersect $Y^z_s$ for all $s \in [0, t]$ and all $z \in \mathscr{A}_t[\gamma] \backslash \{y\}$. Thus, using the fact that $\mathscr{A}_t[\gamma^y] \backslash \{y\} = \mathscr{A}_t[\gamma] \backslash \{y\}$ and applying (i) of Lemma 6, we derive that $Y^z_s[\gamma^y, \omega] = Y^z_s[\gamma, \omega]$ for all $s \in [0, t]$ and all $z \in \mathscr{A}_t \backslash \{y\}$. Consequently, (47) is also true for $\omega$, $y$, and $\gamma^y$. Applying now (ii) and (iii) of Lemma 6, we conclude that the site 0 is occupied by a particle at time $t$ either for the triple $\omega$, $\gamma$, $y$ or for the triple $\omega$, $\gamma^y$, $y$. But given that a site belongs to $\mathscr{A}_t$, it is occupied by an $A$ particle with the probability $\rho/(1-\rho)$ and it is empty with the probability $(1-2\rho)/(1-\rho)$. Thus, we have that

$$\begin{aligned}
&\mathbb{P}[y \text{ is a unique site from } \mathscr{A}_t \text{ such that } \mathscr{B}_y \text{ and } \xi_t(0) \neq 0 \mid \omega, y] \\
&\quad \geqslant (1-\rho)^{-1} \min(\rho, 1-2\rho) \\
&\qquad \times \mathbb{P}[y \text{ is a unique site from } \mathscr{A}_t \text{ such that } \mathscr{B}_y \mid \omega, y] \qquad (48)
\end{aligned}$$

Summing over all $y \in \mathbb{Z}^d$ and integrating over all $\omega \in \Omega$, we derive from (48) that (the first inequality follows from inclusion and the last one is provided by Lemma 7)

$$\begin{aligned}
\mathbb{P}[\xi_t(0) \neq 0] &\geqslant \mathbb{P}[\exists! y \in \mathscr{A}_t : \mathscr{B}_y \text{ and } \xi_t(0) \neq 0] \\
&\geqslant \frac{\min(\rho, 1-2\rho)}{1-\rho} \mathbb{P}[\exists! y \in \mathscr{A}_t : \mathscr{B}_y] \geqslant \frac{(1-\rho)\min(\rho, 1-2\rho)}{8t} \qquad \blacksquare
\end{aligned}$$

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